Game Theory

Lecture 12

Congestion and Potential Games

Introduction

- In general, to represent an n player game in which each player has k actions, we need k^n numbers just to encode the utility functions.
  - Clearly, for even moderately large k and n, nobody could be expected to understand, let alone play rationally, in such a game.
- Hence, we will generally think about games that have substantially more structure -despite being large, they have a concise description that makes them easy to reason about.
- In this lecture, we begin by talking about a class of succinctly representable games, viz. ***congestion games***.
- We will then generalize and introduce a broader but similarly tractable class of games, viz. ***potential games***.
  - Throughout the lecture, it will be convenient to think of players as having cost functions rather than utility functions.
    - Players want to minimize their cost, rather than maximize their utility- but if you like, you can define their utility functions to be the negation of their cost functions.

Congestion Games

Definition *A congestion game is defined by:*

1. *A set of n players P*
2. *A set of m facilities F*
3. *For each player i, a set of actions A_i. Each action $a_i \in A_i$ represents a subset of the facilities: $a_i \subseteq F$.*
4. *For each facility $j \in F$, a cost function $\ell_j : \{0, \ldots, n\} \to \mathbb{R}_{>0}$.*

$\ell_j(k)$ represents "the cost of facility j when k players are using it".

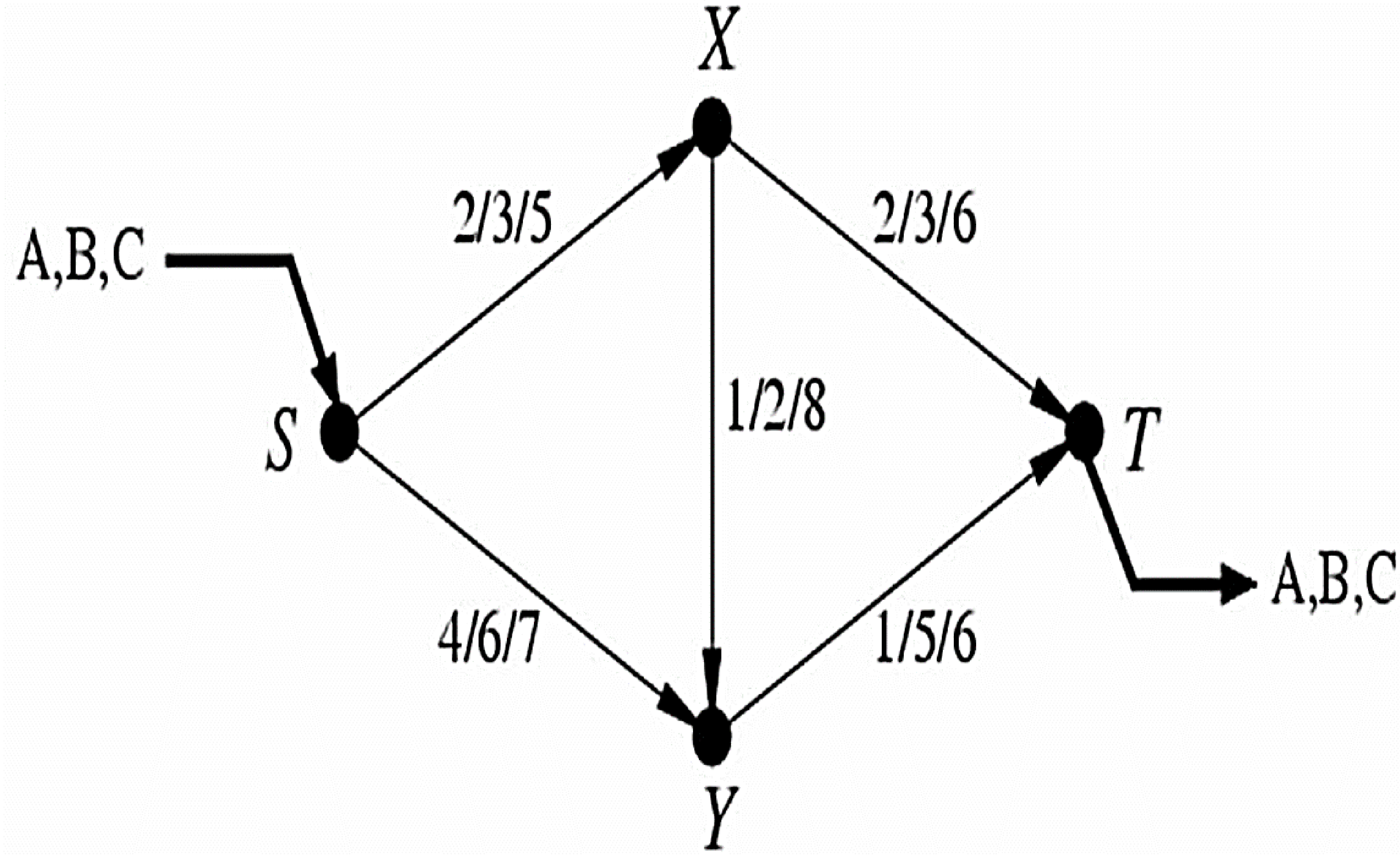
Player costs are then defined as follows. For action profile $a = (a_1, \ldots, a_n)$ define $n_j(a) = |\{i : j \in a_i\}|$ to be the number of players using facility j. Then the cost of agent i is:

$$c_i(a) = \sum_{j \in a_i} \ell_j(n_j(a))$$

i.e. the total cost of the facilities she is using.

Congestion Games: Example

- Let's give an illustrative example of a congestion game.
- Players A,B and C have to go from point *S* to *T* using road segments *SX*,*XY*,...etc. (See Figure).



- Numbers on edges denote the cost for a single user for using the corresponding road segment, where the actual cost is a function of the actual number of players using that road segment (i.e. a discrete delay function).
  - For example: if segment SX is used by 1,2, or 3 users, the cost on that segment would be 2,3, or 5, respectively. The total cost for a player is the sum of the costs on all segments he uses.

Finding Equilibria in Congestion Games

- Ok -so congestion games define an interesting class of *n*-player, many-action games that nevertheless have a simple structure and concise representation. What can we say about them?
  - Do they have pure strategy Nash equilibria?
  - Can we find those equilibria efficiently?
  - Would agents, interacting together in a decentralized way naturally find said equilibria?
- To answer many of these questions, we will consider "***Best response dynamics***".
  - We present it as an algorithm, but you could equally well think about it as a natural model for how people would actually behave in a game.
  - The basic idea is this: we start with players playing an arbitrary set of actions. Then, in arbitrary order, they take turns changing their actions so that they are best responding to their opponents. We continue until (if?) this process converges.

Best Response Dynamics

Algorithm 1 Best Response Dynamics

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i such that $a_i \notin \arg\min_{a \in A_i} c_i(a, a_{-i})$ **do**
  Set $a_i = \arg\min_{a \in A_i} c_i(a, a_{-i})$
end while
Halt and return a.

Claim *If best response dynamics halts, it returns a pure strategy Nash equilibrium.*

Proof Immediate from halting condition- by definition, every player must be playing a best response. ■

- Of course, it won't always halt – e.g., consider matching pennies -but what the above claim means is that to prove the existence of pure strategy Nash equilibria in congestion games, it suffices to analyze the above algorithm and prove that it always halts.

Theorem *Best response dynamics always halt in congestion games.*

Corollary *All congestion games have at least one pure Nash equilibrium.*

Proof We will study the following potential function $\phi : A \to \mathbb{R}$ defined as follows:

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

(Note that the potential function is not social welfare).

Now consider how the potential function changes in a single round of best response dynamics, when player i switches from playing some action $a_i \in A_i$ to playing $b_i \in A_i$ instead.

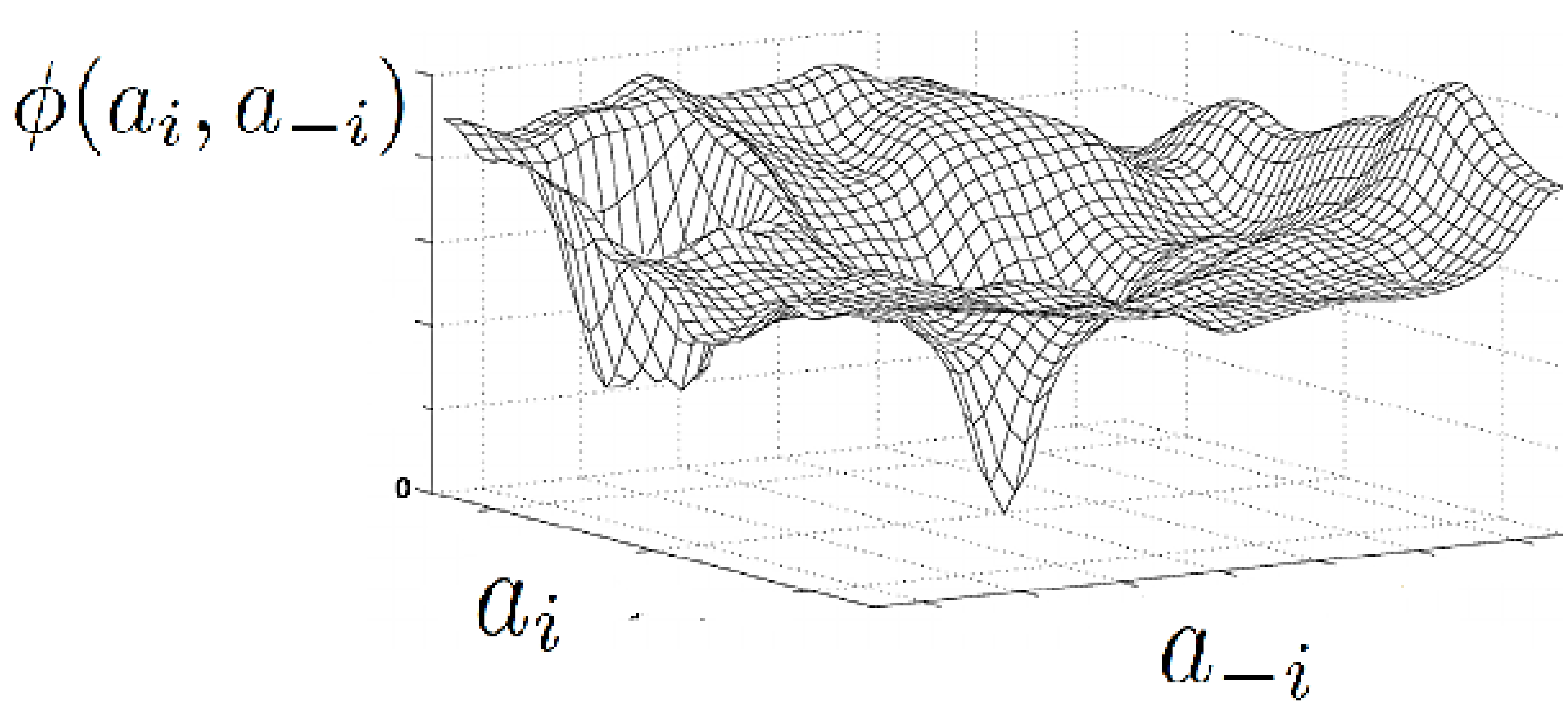
First, because this was a step of best response dynamics, we know that the switch decreased player i's cost:

$$\begin{aligned}
\Delta c_i &\equiv c_i(b_i, a_{-i}) - c_i(a_i, a_{-i}) \\
&= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a) + 1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(a)) \\
&< 0
\end{aligned}$$

The change in potential is:

$$\begin{aligned}\Delta\phi &\equiv \phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a)+1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(a)) \\ &= \Delta c_i\end{aligned}$$

- Hence, we know that $\Delta\phi < 0$.
- But since ϕ can take on only finitely many different values (why?) and decreases between each round of best response dynamics, the potential function will eventually reach a local minima. At this point, no player can achieve any improvement, therefore, best response dynamics must eventually halt; i.e., we reach a pure strategy Nash equilibrium. ❑



Speeding up the Convergence

- Of course, we have only proven convergence, not fast convergence.
  - It might take a long time, and if it takes an unreasonably long time (say exponentially many rounds in the number of players), then it might not be a reasonable prediction to assert that rational players will play a Nash equilibrium.
  - <u>Bad news:</u> it might take a really long time for best response dynamics to converge!
  - But we will be able to say that they converge quickly to an approximate Nash equilibrium.

Definition *An action profile* $\boldsymbol{a} \in \boldsymbol{A}$ *is an "*$\boldsymbol{\varepsilon}$ ***−approximate****" pure NE if for every player i, and for every action* $a_i' \in A_i$*:*

$$c_i(a_i, a_{-i}) \leq c_i(a_i', a_{-i}) + \epsilon$$

i.e. nobody can gain more than ε by deviating.

Finding Approximate NE

- Lets consider a modification of best response dynamics that only has people move if they can decrease their cost by at least ε:

Algorithm 2 FindApproxNash(ϵ)

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i, a'_i such that $c_i(a'_i, a_{-i}) \leq c_i(a_i, a_{-i}) - \epsilon$ **do**
  Set $a_i = \arg\min_{a \in A_i} c_i(a, a_{-i})$
end while
Halt and return a.

Claim *If FindApproxNash(ε) halts, it returns an ε-approximate pure strategy Nash equilibrium.*

Proof Immediate by definition. ❑

Congestion Games

Theorem *In any congestion game, FindApproxNash(ϵ) halts after at most:*

$$\frac{n \cdot m \cdot c_{max}}{\epsilon}$$

Proof We revisit the potential function ϕ. Recall that $\Delta c_i = \Delta\phi$ on any round when player I moves.

Observe also that at every round, $\phi \geq 0$, and

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k) \leq n \cdot m \cdot c_{max}$$

By definition of the algorithm, we have $\Delta c_i = \Delta\phi \leq -\epsilon$ at every round, and so the theorem follows. ■

A Natural Generalization: Weighted Congestion Games

- Weighted congestion games look almost like a congestion game (in that there are still players and facilities), but the costs of each facility depend not just on how many people are playing on it, but on which players are playing on it.

First Example: A Load Balancing Game

Definition *A* load balancing game on identical machines *models n players $i \in P$ scheduling jobs of size $w_i > 0$ on m identical machines F. The game has:*

1. *Action space $A_i = F$ for each player*
2. *For each machine $j \in F$, a load $\ell_j(a) = \sum_{i:a_i=j} w_i$*

The cost of player i is the load of the machine he plays on: $c_i(a) = \ell_{a_i}(a)$.

Theorem *Best response dynamics converge in load balancing games on identical machines.*

Corollary *Load balancing games on identical machines have pure strategy Nash equilibria*

Proof We use a variation of our potential function argument (but need a new potential function).

Define $\phi(a) = \frac{1}{2}\sum_{j=1}^{m} \ell_j(a)^2$.

Suppose player i switches from machine j to machine j'.

Then we have:

$$\begin{aligned}
\Delta c_i(a) &\equiv c_i(j', a_{-i}) - c_i(j, a_{-i}) \\
&= \ell_{j'}(a) + w_i - \ell_j(a) \\
&< 0
\end{aligned}$$

Similarly, we have:

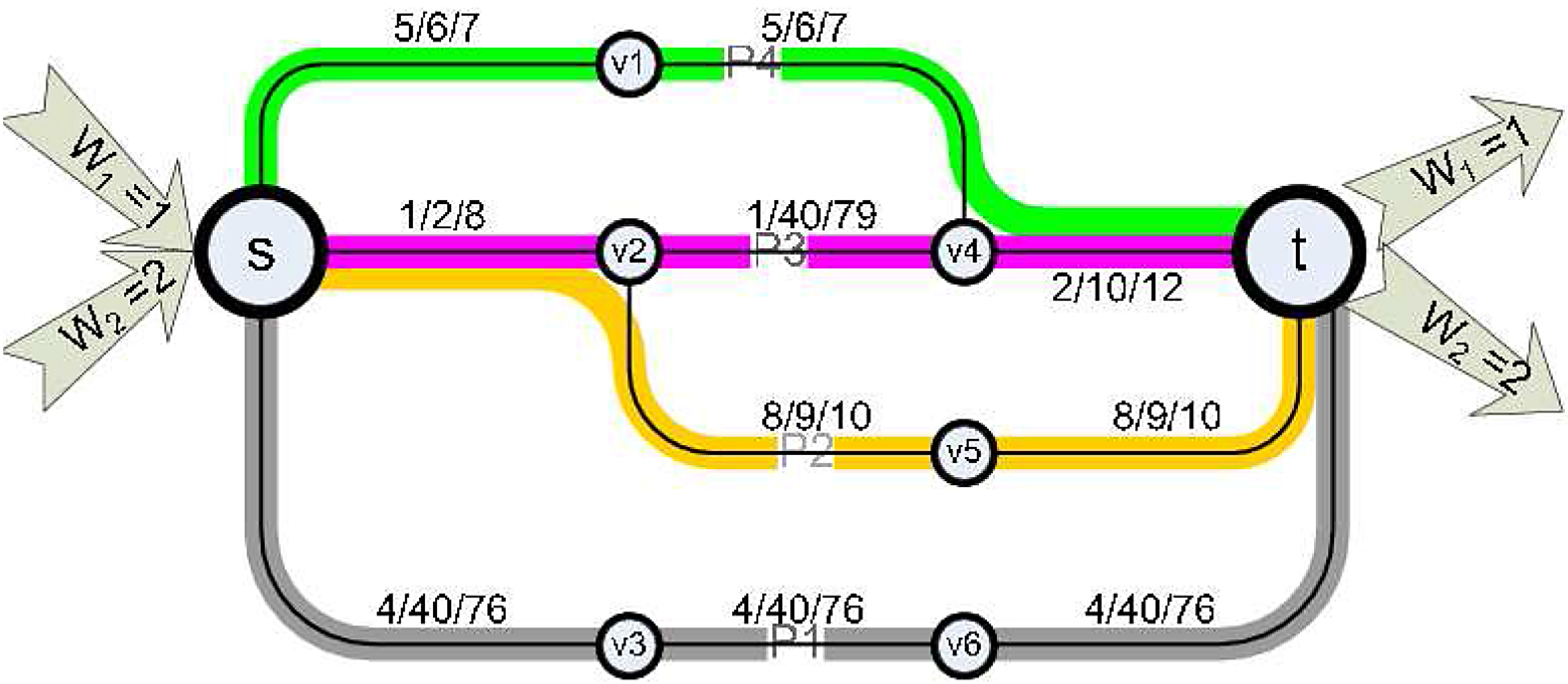
$$\begin{aligned}
\Delta\phi(a) &\equiv \phi(j', a_{-i}) - \phi(j, a_{-i}) \\
&= \frac{1}{2}\left((\ell_{j'}(a) + w_i)^2 + (\ell_j(a) - w_i)^2 - \ell_{j'}(a)^2 - \ell_j(a)^2\right) \\
&= \frac{1}{2}\left(2w_i\ell_{j'}(a) + w_i^2 - 2w_i\ell_j(a) + w_i^2\right) \\
&= w_i\left(\ell_{j'}(a) + w_i - \ell_j(a)\right) \\
&= w_i \cdot \Delta c_i(a) \\
&< 0
\end{aligned}$$

- Note that unlike in congestion games, the change in potential function is not equal to the change in player cost when player's make unilateral deviations.
- Nevertheless, it decreases with every better-response deviation, and because it is always non-negative (and because there are only finitely many action profiles), this process must eventually halt.

Second Example of a Weighted Congestion Game

Two Players wish to choose a route $s-t$, each has a route $s-t$, each has a weight $w_1 = 1, w_2 = 2$.

The edge's discrete delay functions are as shown in the figure.



A necessary condition for a pure equilibrium is that each player chooses a route that is in his $BestResponse$ given the other player's chosen route. That is, $a_1 \in BR_1(a_2)$ and $a_2 \in BR_2(a_1)$

In this example there are only four $s-t$ routes, and by going over all 4 options for a_i it is easy to see that the two necessary conditions can not hold at the same time, and therefore in this example there is no pure equilibrium.

Potential Games

- We saw that congestion games always have pure NE, but weighted congestion games do not necessarily have pure NE.
- On the other hand, not all games in which best response dynamics converge need to "look like" congestion games.
- We ask how far we can go beyond congestion games while still being certain that best response dynamics will converge to pure strategy Nash equilibria.
  - ➢ Recall that if BRD converges, it is necessarily to a pure strategy Nash equilibrium, so the question is really just *when do they converge.*
- We will see that the class of games in which best response dynamics converge to a pure NE extends beyond congestion games, and similarly, beyond games in which we can define a potential function that changes exactly as the best response player's utility changes (e.g. the load balancing game).
- We start with introducing ***exact potential games*** (EPG):

Definition *A function $\phi : A \to \mathbb{R}_{>0}$ is an* exact potential function *for a game G if for all $a \in A$, all i, and all $a_i, b_i \in A_i$:*

$$\phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) = c_i(b_i, a_{-i}) - c_i(a_i, a_{-i})$$

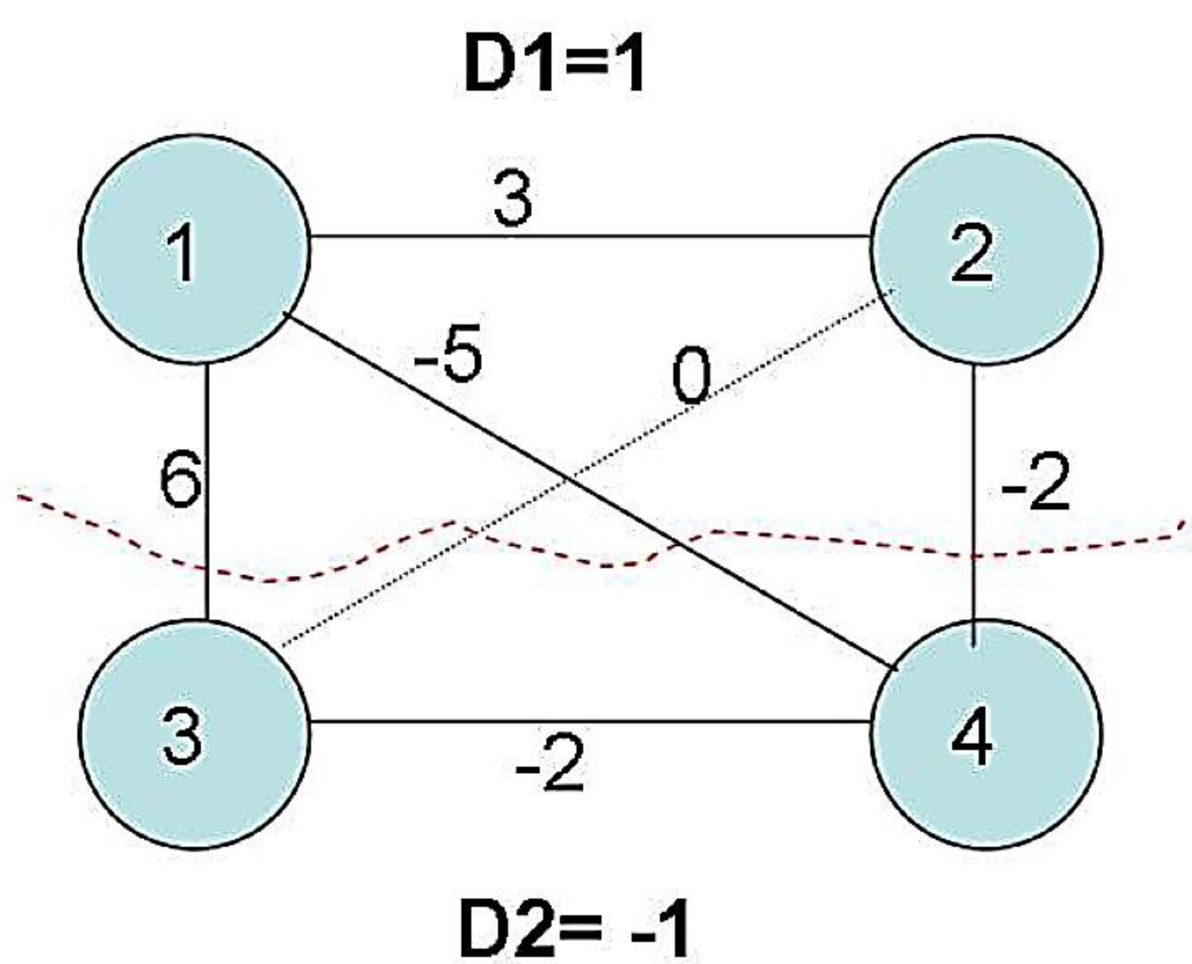
Definition A game G is called an *exact potential game (EPG)* if it has an exact potential function.

Example of an Exact Potential Game

Consider an undirected graph $G = (V, E)$ with a weight function $\vec{\omega}$ on its edges.

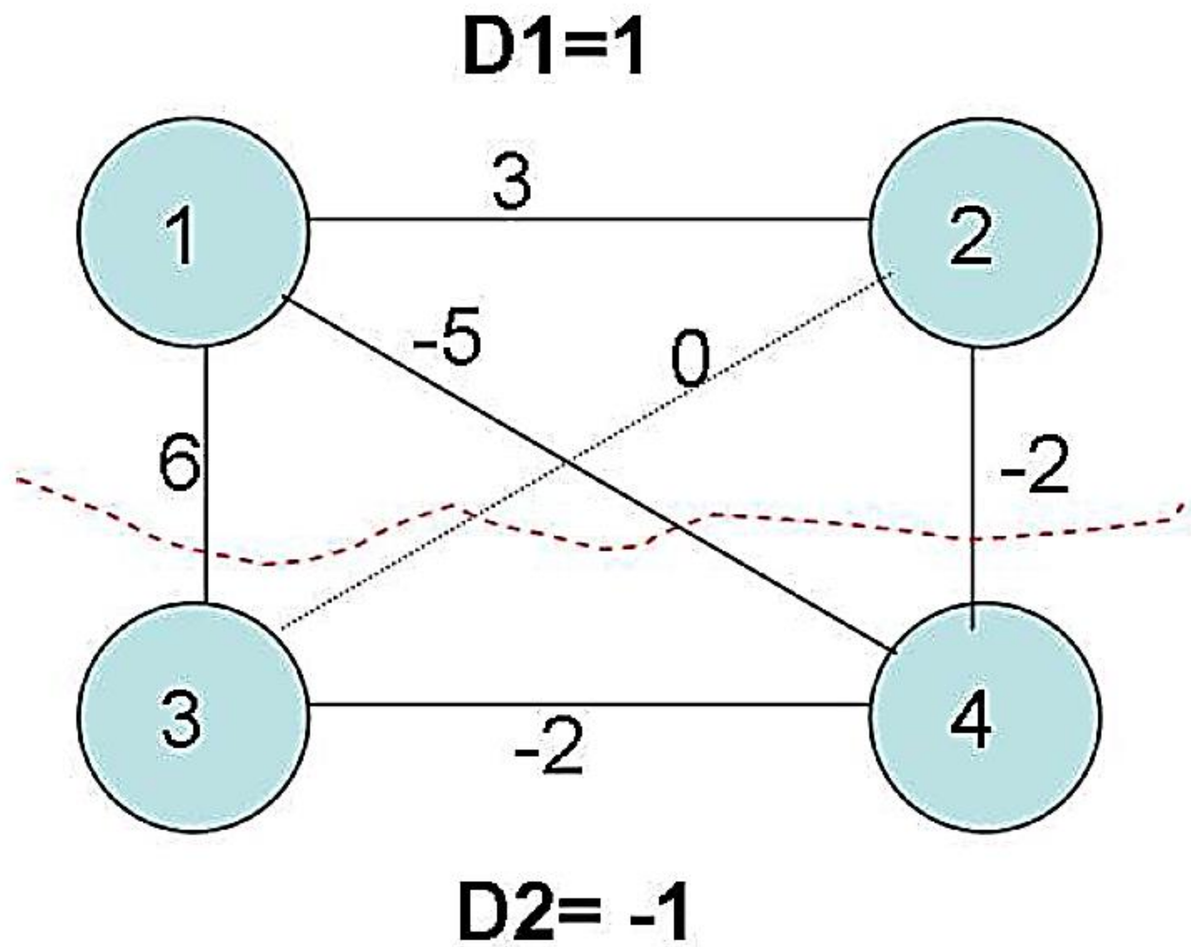
In this game the players are the vertices and the goal is to partition the vertices set V into two distinct subsets D_1, D_2 (where $D_1 \cup D_2 = V$):

For every player i, choose $s_i \in \{-1, 1\}$ where choosing $s_i = 1$ means that $i \in D_1$ and $s_i = -1$ means that $i \in D_2$.



The weight on each edge denotes how much the corresponding vertices 'want' to be on the same set.

Example of an Exact Potential Game



Thus, define the value function of player i as

$$u_i(\vec{s}) = \sum_{j \neq i} \omega_{i,j} s_i s_j.$$

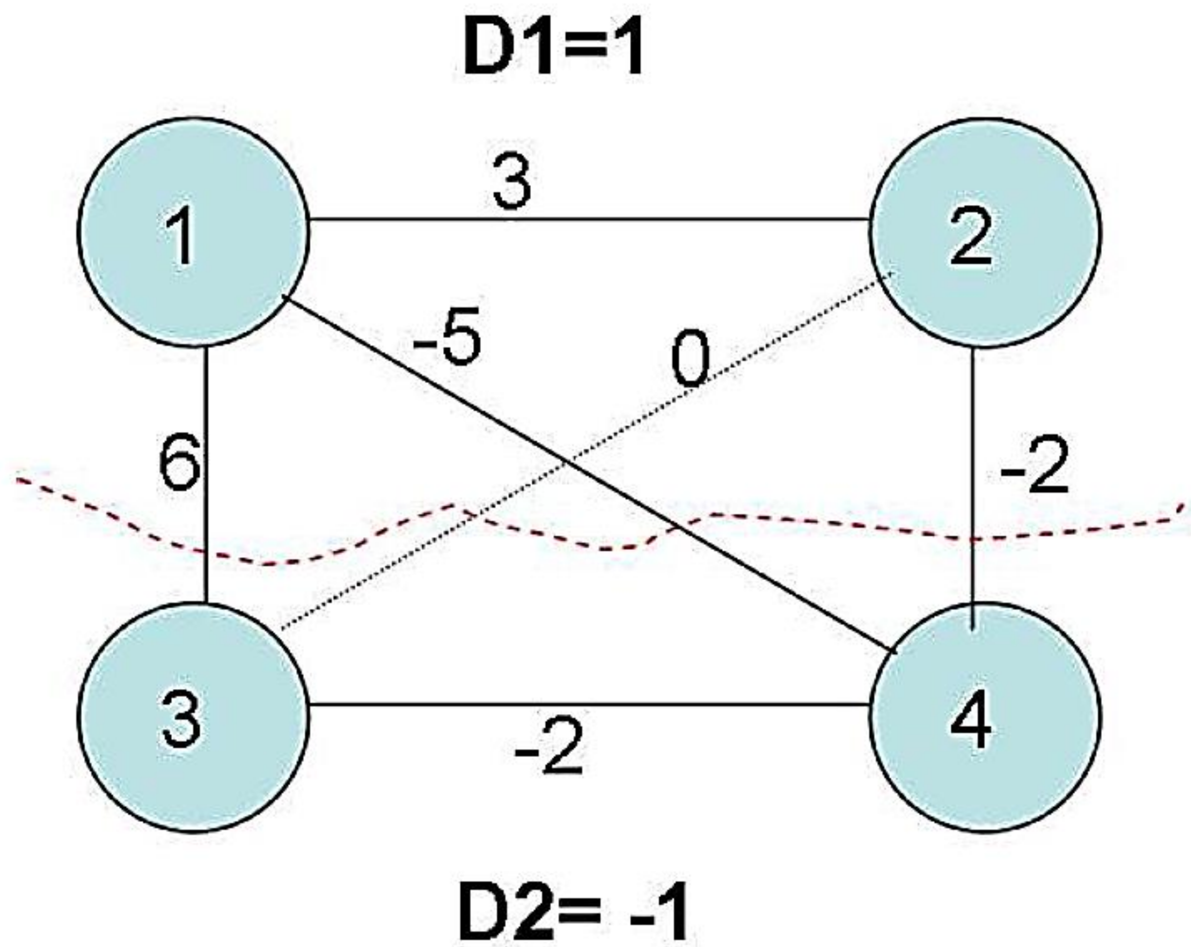
A player 'gains' $\omega_{i,j}$ for players that are in the same with him, and 'loses' for player in the other set.

Note that $\omega_{i,j}$ can be negative.

Each player tries to maximize its utility function.

On the example given in the figure, it can be seen that players 1,2 and 4 have no interest in changing their strategies, however, player 3 is not satisfied, it can increase his profit by changing his set to D_1.

Example of an Exact Potential Game



Using $\Phi(\vec{s}) = \sum_{j<i} \omega_{i,j} s_i s_j$ as our potential function, let us

consider the case where a single player i changes its strategy (shifts from one set to another):

$$\Delta u_i = \sum_{j \neq i} \omega_{i,j} s_i s_j - \sum_{j \neq i} \omega_{i,j} (-s_i) s_j =$$

$$2 \sum_{j \neq i} \omega_{i,j} s_i s_j = \Delta(\Phi)$$

Which means that Φ is an exact potential function, therefore we conclude that the above game is an exact potential game.

Some Remarks on Exact Potential Games

- Any congestion game (as defined earlier) is an exact potential game.
- Having an exact potential function is <u>sufficient</u> for best response dynamics to converge.
- However, it is not clear that it is <u>necessary</u> {in particular, we did not exhibit one for the load balancing game (See slides 12,13)}.

Definition $\phi : A \to \mathbb{R}_{\geq 0}$ *is an* ordinal potential function *for a game G if for all $a \in A$, all i, and all $a_i, b_i \in A_i$:*

$$\boxed{(c_i(b_i, a_{-i}) - c_i(a_i, a_{-i}) < 0) \Rightarrow (\phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) < 0)}$$

i.e. the change in utility is always equal in sign *to the change in potential.*

- We will now show that ordinal potential functions exactly characterize those games in which best response dynamics is guaranteed to converge.

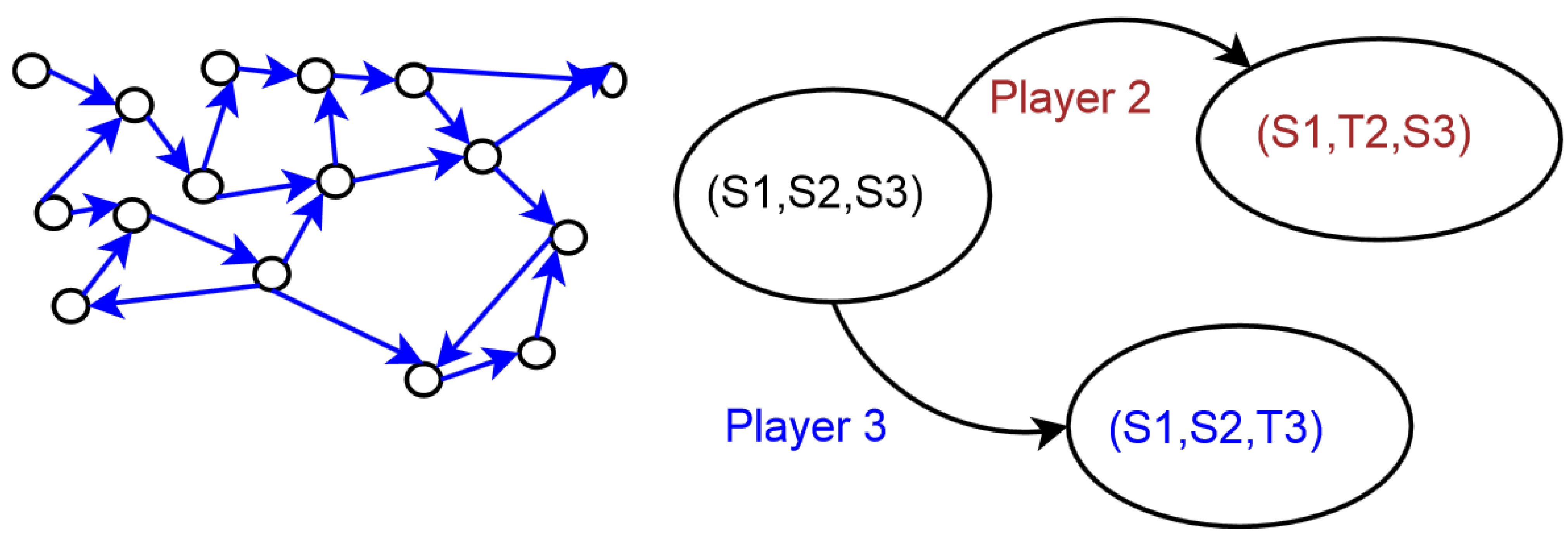
Definition A game G is called an *ordinal potential game (OPG)* if it has an ordinal potential function.

Theorem *Best response dynamics is guaranteed to converge in a game G* ***<u>if and only if</u>*** *the game is an ordinal potential game.*

Proof We already know how to show having an ordinal potential function is sufficient for the convergence of best response dynamics {this is the template proof we have applied 3 times now.} To prove that its existence is necessary, consider the following ***state graph*** G = (V;E):

1. Let each $a \in A$ be a vertex in the graph: i.e. $V = A$.
2. For each pair of vertices $a, b \in V$, add a directed edge (a, b) if it is possible to get from b to a via a best response move – i.e. if there is some index i such that $b = (b_i, a_{-i})$, and $c_i(b_i, a_{-i}) < c_i(a)$.

state graph $\mathcal{G} = (V, E)$



V = **states** E = **better/best responses**

- Best response dynamics can be viewed as traversing this graph, starting at some arbitrary vertex *a*, and then traversing the graph along its edges (which it can do breaking ties arbitrarily).
- The Nash equilibria are exactly the sinks in this graph (in which no player can make a best response move).
- If best response dynamics always converges, it must be that the graph has no cycles.

In this case, we construct an ordinal potential function ϕ as follows.

Since the graph has no cycles there must be reachable from every state a some some sink s (i.e. a pure strategy Nash equilibrium)

For each state a, let $\phi(a)$ denote the length of the longest finite path in G from a to any sink s.

The property we require is that $\phi(b) < \phi(a)$ for any pair of vertices (a, b) with an edge $a \to b$.

But observe that by definition, if there is an edge $a \to b$, then

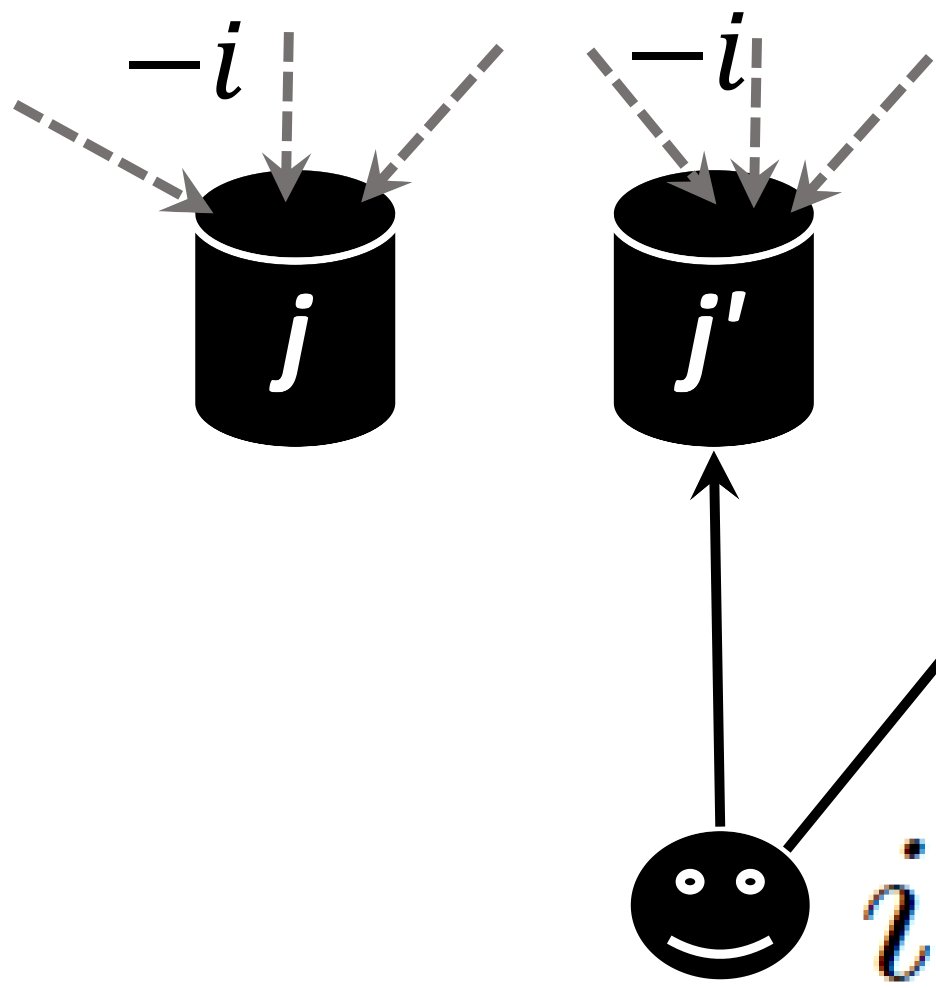
$$\phi(a) \geq \phi(b) + 1$$

because there is at the very least a path that first goes to b, and then takes the longest path from b to a sink, which completes the proof. ■

Appendix

$$c_i(a_i, a_{-i}) =$$

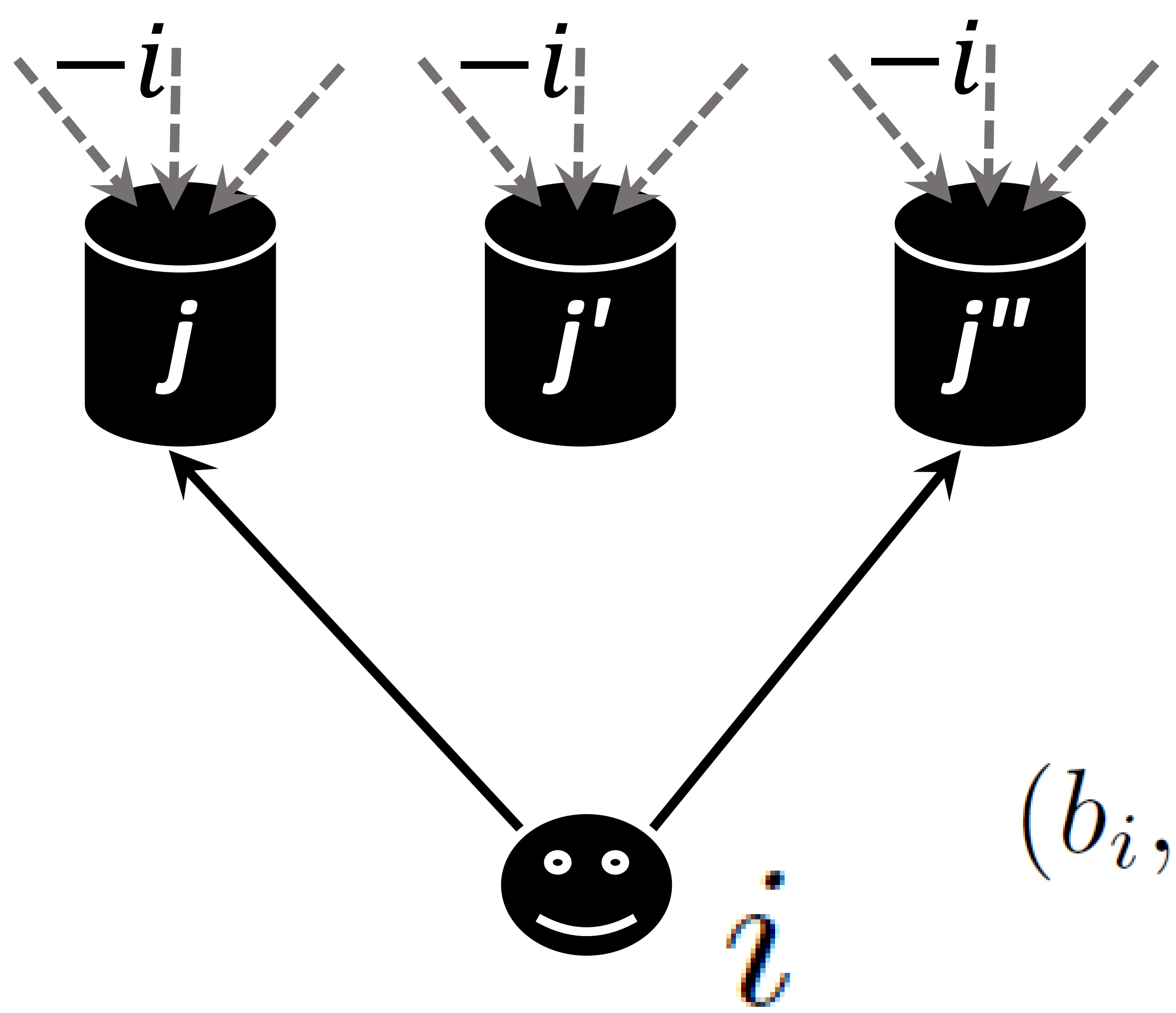
$$\ell_{j'}\left(n_{j'}(\boldsymbol{a})\right) + \ell_{j''}\left(n_{j''}(\boldsymbol{a})\right)$$



(a_i, a_{-i})

$$c_i(b_i, a_{-i}) =$$

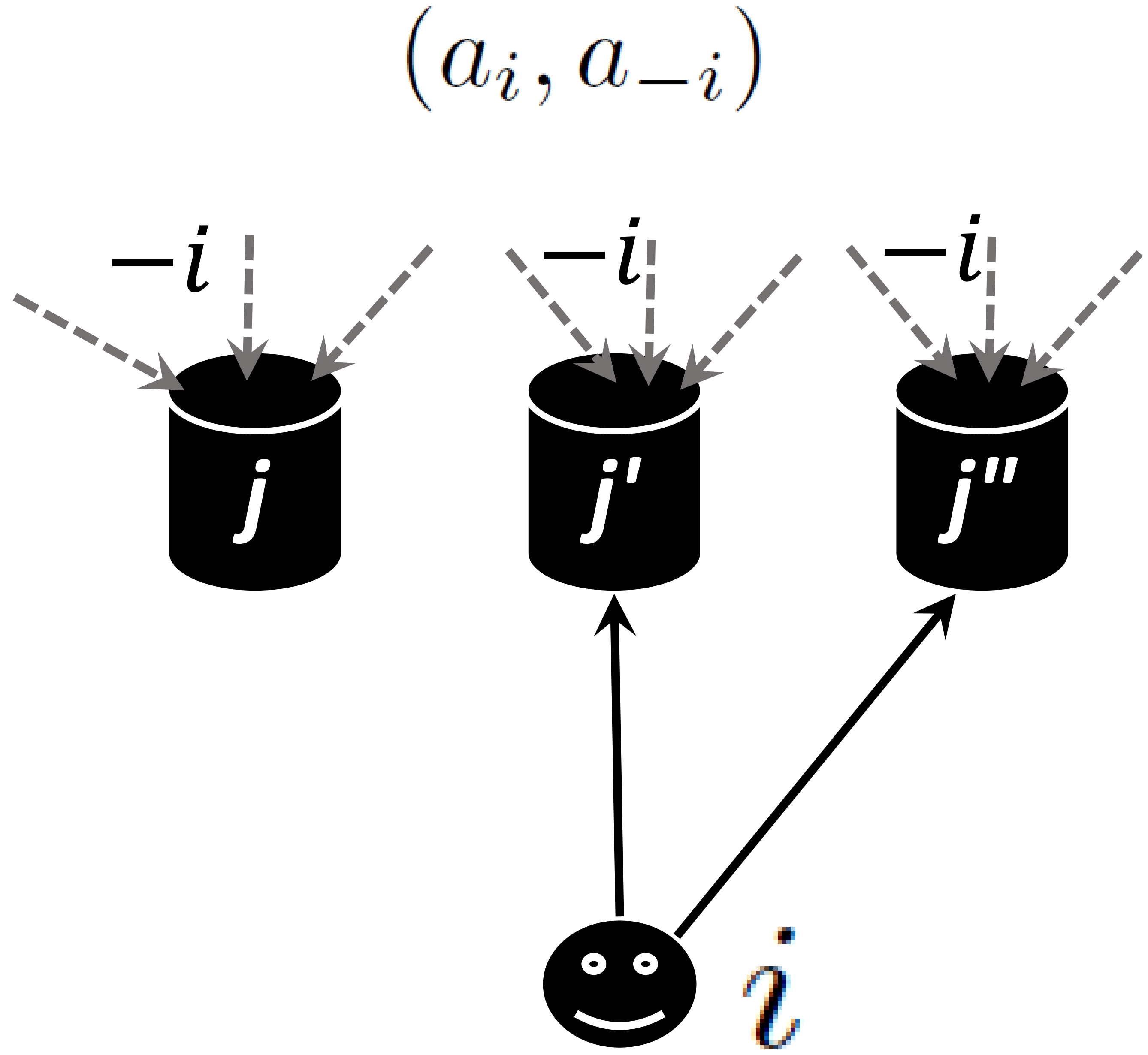
$$\ell_j\left(n_j(\boldsymbol{a}) + 1\right) + \ell_{j''}\left(n_{j''}(\boldsymbol{a})\right)$$



(b_i, a_{-i})

$$\begin{aligned} \Delta c_i &\equiv c_i(b_i, a_{-i}) - c_i(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a) + 1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(a)) \\ &< 0 \end{aligned}$$

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

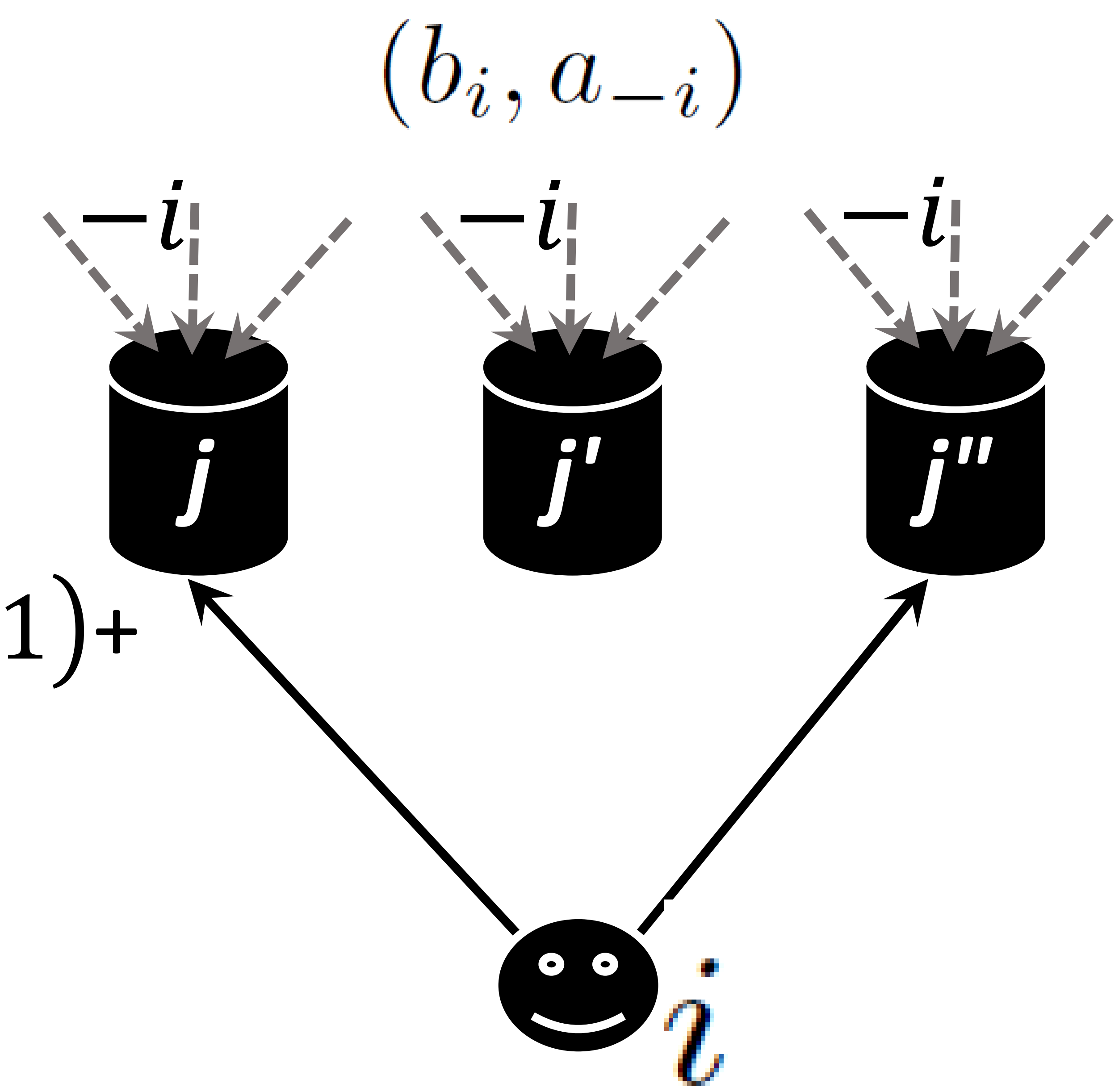


$\phi(a_i, a_{-i}) =$

$0 + \ell_j(1) + \cdots + \ell_j\left(n_j(\boldsymbol{a})\right) +$

$0 + \ell_{j'}(1) + \cdots + \ell_{j'}\left(n_{j'}(\boldsymbol{a})\right) +$

$0 + \ell_{j''}(1) + \cdots + \ell_{j''}\left(n_{j''}(\boldsymbol{a})\right)$



$\phi(b_i, a_{-i}) =$

$0 + \ell_j(1) + \cdots + \ell_j\left(n_j(\boldsymbol{a})\right) + \ell_j\left(n_j(\boldsymbol{a}) + 1\right) +$

$0 + \ell_{j'}(1) + \cdots + \ell_{j'}\left(n_{j'}(\boldsymbol{a}) - 1\right) +$

$0 + \ell_{j''}(1) + \cdots + \ell_{j''}\left(n_{j''}(\boldsymbol{a})\right)$

$$\begin{aligned}
\Delta\phi &\equiv \phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) \\
&= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a) + 1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(a)) \\
&= \Delta c_i
\end{aligned}$$